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Extended relativity, *T*-violation and a new kind of particle

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Abstract. The problem of minimal extensions of the Lorentz group by superluminal-like transformations is analysed. It is found that the 'extended relativity groups' are closely related to the complex Poincaré group.

'Extended relativity' is then studied as a continuous kinematical symmetry of physical states (not necessarily superluminal), and its quantum-mechanical implications obtained by the construction of some classes of irreducible unitary representations. It is found that, if there are massive or massless states having 'extended relativity' as their kinematical symmetry group, there is a superselection rule operating between these states and ordinary matter. Their mutual interactions may display a new type of spontaneous breaking of PT invariance.

The possible relevance of these new hypothetical states to T violation is neutral kaon decays and their observation as real states is discussed in two simple model examples.

1. Introduction

Motivated by the problem of whether faster-than-light propagation has any physical relevance, a number of authors (e.g., Recami and Mignani 1974 and references therein) have discussed the extension of the Lorentz group to a group of 'extended relativity' containing transformations that might be interpreted as coordinate transformations between subluminal and superluminal frames. In the original motivation of these studies such superluminal frames would be the rest frames of tachyons. A formalism is thus developed to deal with these hypothetical states which is alternative and not necessarily equivalent to the conventional view (Bilaniuk *et al* 1962) that tachyons (like photons) are created and destroyed but never brought to rest and that the set of physically equivalent inertial frames is restricted to those in slower-than-light uniform notion.

Whether or not 'extended relativity' is relevant to tachyons and whether or not tachyons are relevant to physics, the extended groups may still be interesting in their own right.

Lorentz transformations and space–time translations (i.e. the Poincaré group) together with general principles such as microscopic causality and positivity form the basic foundation for most of our understanding of subnuclear phenomena. If, as the work in 'extended relativity' suggests, the kinematical symmetry group of nature may be enlarged, at least in some subspaces of the physical space, this may have very far-reaching consequences. An enlargement of something as fundamental as the kinematical symmetry group may have consequences not only on eventual hypothetical superluminal states but it may also open up some new possibilities in the domain of the less exotic luminal and subluminal states. This defines the point of view adhered to in

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this paper. Our purpose will be to study the transformations of extended relativity as a symmetry group, exploring some of its consequences in the domain of massive and massless states.

That the kinematical symmetry group might not be the same for all particles should not be suprising if one remembers that the parity operation is not implemented for neutrinos. Conversely, one might imagine that some particles possess a kinematical group larger than the usual Poincaré group (plus the discrete transformations).

To make the paper reasonably self-contained, in § 2 we review the derivation of the extended relativity groups from the usual postulates of isotropy of space, equivalence of inertial frames and homogeneity, including in the set of possible transformations those with a velocity greater than the invariant one. We find that there are two distinct minimal ways to construct the extended relativity group and later on we show that for the purposes of this paper these have identical consequences.

Using the results compiled in appendix 1, the close relation between these extended relativity groups and the complex Lorentz group is exhibited in § 3.

In § 4 a particularly simple class of unitary representations of the 'extended relativity group' is derived; the symmetry-breaking properties, superselection rules and possible relevance to T-violation of this class are explored in § 5.

2. Extended relativity groups

Let a^{μ}_{ν} denote the matrix elements of a coordinate transformation between two equivalent inertial frames, $x'^{u} = a^{\mu}_{\nu} x^{\nu}$.

Linearity follows from the homogeneity of space–time. With isotropy of space and equivalence of inertial frames one deduces the existence of an invariant quantity K with dimensions of an inverse squared velocity (Berzi and Gorini 1969, Gorini and Zecca 1970):

$$\frac{1}{u^2}\left(1-\frac{1}{a^0_0(u)a^0_0(-u)}\right)=K=\frac{1}{c^2} \tag{2.1}$$

where $a^0_0(u)$ is a matrix element of a boost. From equation (2.1) it follows that

$$a^0_0(u)a^0_0(-u)=\frac{1}{1-u^2/c^2}. \tag{2.2}$$

Denoting by 1 the direction of the boost, one also deduces from the same principles that $a^1_1(u)=a^0_0(u)$, $a^1_0(u)=-ua^1_1(u)$; $a^0_1(u)=(-u/c^2)a^0_0(u)$.

From (2.2) one finds two distinct types of transformation, those for which $u^2/c^2<1$ and those for which $u^2/c^2>1$. They cannot be connected by continuous variations of the real parameter u because of the singularity at the 'light barrier' ($u^2/c^2=1$).

For $u^2/c^2<1$ (the case of special relativity) the matrices $\mathbf{a}(u)$ and $\mathbf{a}(-u)$ are transformations between identical slower-than-light (bradyonic) frames; therefore isotropy requires $a^0_0(u)=a^0_0(-u)$ and one obtains

$$a^0_0(u)=(1-u^2/c^2)^{-1/2}$$

$$a^0_1(u)=-\frac{u}{c^2}(1-u^2/c^2)^{-1/2}.$$

For $u^2/c^2>1$ and in contrast to the previous case, $a_0^0(u)$ and $a_0^0(-u)$ refer to transformations of a different nature, because if the first is from a bradyonic to a tachyonic frame (BT transformation), the second is from a tachyonic to a bradyonic frame (TB transformation). Therefore whereas isotropy would require $a_0^0(u)_{\mathrm{BT}}=a_0^0(-u)_{\mathrm{BT}}$ it is not obvious that it must also require $a_0^0(u)_{\mathrm{BT}}=a_0^0(-u)_{\mathrm{TB}}$.

One could somehow justify this last equality by insisting that all frames, bradyonic and tachyonic, should be equivalent, because after all if S is a bradyonic for S and S' tachyonic for S then S will be tachyonic for S' and S' bradyonic for S'. Therefore there is no invariant way to decide whether a particular transformation is of the BT or of the TB type, and the natural choice would still be $a_0^0(u)=a_0^0(-u)$. Under these circumstances it follows from equation (2.2) that $a_0^0(u)$ is purely imaginary and the final result would be

$$\begin{aligned} t' &= \frac{\mathrm{i}t}{(u^2/c^2-1)^{1/2}} - \frac{\mathrm{i}x^{\mathrm{L}}(u/c^2)}{(u^2/c^2-1)^{1/2}} \\ x_{\mathrm{L}}' &= \frac{-\mathrm{i}ut}{(u^2/c^2-1)^{1/2}} + \frac{\mathrm{i}x^{\mathrm{L}}}{(u^2/c^2-1)} \\ \boldsymbol{x}^{\mathrm{T}\prime} &= \boldsymbol{x}^{\mathrm{T}} \end{aligned} \tag{2.3}$$

where the indices L and T denote longitudinal and transverse components and one of the signs of the square root was arbitrarily chosen. From the relation

$$(ct')^2-|\boldsymbol{x}'|^2=[(ct)^2-|\boldsymbol{x}|^2]\frac{a_0^0(u)}{a_0^0(-u)}, \tag{2.4}$$

which has general validity, it follows that in the case of equations (2.3) the four-dimensional space–time interval remains strictly invariant for superluminal transformations.

If one had required that for superluminal transformations time-like intervals be transformed into space-like intervals and vice versa, as one would have intuitively expected to happen for real transformation matrices, one should have chosen instead $a_0^0(u)=-a_0^0(-u)$ and in this case the result would have been

$$\begin{aligned} t' &= \frac{t}{(u^2/c^2-1)^{1/2}} - \frac{x^{\mathrm{L}}(u/c^2)}{(u^2/c^2-1)^{1/2}} \\ x^{\mathrm{L}\prime} &= \frac{-ut}{(u^2/c^2-1)^{1/2}} + \frac{x^{\mathrm{L}}}{(u^2/c^2-1)^{1/2}} \\ \boldsymbol{x}^{\mathrm{T}\prime} &= -\mathrm{i}x^{\mathrm{T}} \end{aligned} \tag{2.5}$$

with the imaginary matrix element in the transverse component transformation being required by equation (2.4).

These were, in fact, the superluminal transformations considered by Recami and Mignani (1974) and they differ from the previous ones by an overall mutiplication by $-\mathrm{i}$.

It is only in the bi-dimensional (one space, one time) case that the matrices of the extended group can be purely real. In such a case (Parker 1969) it would seem natural to choose the transformations of equations (2.5) instead of those of equations (2.3).

For space–time dimension greater than two, however, both transformation groups have to be complex and the interpretation of intervals as space-like or time-like becomes a delicate matter related to the actual metric machinery associated with each frame (Mignani and Recami 1974, Goldoni 1972). Therefore both sets of equations, (2.3) or (2.5), seem to be acceptable as superluminal extensions of special relativity.

If one actually wants to speculate about the physical meaning of tachyons and superluminal frames the interpretation of the above transformations may be different in each case; however, for our more modest purpose of exploring the consequences of extended relativity as a symmetry group for luminal and subluminal phenomena, their content, as we will see shortly, is essentially identical.

The extended relativity groups are the multiplicative closures of the set of superluminal boosts and the restricted Lorentz group. Their close relation to the complex Lorentz group will be derived in the next section.

3. Extended relativity and the complex Lorentz group

In this section we use the results and notation of appendix 1 and natural units such that $c = 1$.

Let us consider first the case of the transformations of equations (2.3). Consider the limit when $u^2 \to \infty$, with $u > 0$, of a superluminal transformation (of the form of equations (2.3)) along the direction i. The resulting transformation S_i has the properties

$$\Lambda_i(u) = S_i \Lambda_i(1/u) \qquad \text{for } u > 0, |u| > 1$$

and

$$\Lambda_i(u) = S_i^2 S_i \Lambda_i(1/u) \qquad \text{for } u < 0, |u| > 1,$$

i.e. any superluminal velocity transformation $\Lambda_i(u)$ can be obtained from a transformation of the real restricted Lorentz group multiplied by a power of S_i.

The problem of finding the multiplicative closure of the superluminal transformations and the restricted Lorentz group $L_+^\uparrow$ is thus reduced to the simpler one of finding the closure of the set $\{S_i, L_+^\uparrow\}$. Like any one of the transformations defined by equations (2.3), S_i is a complex Lorentz matrix and, in terms of the generators defined in appendix 1, its representation is

$$S_i = \exp(-\mathrm{i}\tfrac{1}{2}\pi H_i). \tag{3.1}$$

From this one finds

$$S_i^{-1} \exp(-\mathrm{i}\theta J_j) S_i = \epsilon_{ijk} \exp(\mathrm{i}\theta H_k) + \delta_{ij} \exp(-\mathrm{i}\theta J_j) \tag{3.2a}$$

$$S_i^{-1} \exp(-\mathrm{i}u K_j) S_i = \epsilon_{ijk} \exp(\mathrm{i}u I_k) + \delta_{ij} \exp(-\mathrm{i}u K_j). \tag{3.2b}$$

Notice that $S_i^{-1} = S_i^3$.

From these results one concludes immediately that the closure of S_i and $L_+^\uparrow$ is $L_+(\mathrm{C})$. Therefore in this case the group of extended relativity is nothing else but the complex Lorentz group.

Let us now consider what happens if for the definition of superluminal velocity transformations one uses equations (2.5) instead of equations (2.3). In this case

$$\Lambda_i'(u) = -\mathrm{i}S_i \Lambda_i'(1/u) \qquad u > 0, |u| > 1$$

$$\Lambda_i'(u) = \mathrm{i}S_i \Lambda_i'(1/u) \qquad u < 0, |u| < 1.$$

Therefore the extended relativity group would in this case be the multiplicative closure of $\{\pm \mathrm{i}S_i, L_+^\uparrow\}$.

The inverse of $-\mathrm{i}S_i$ is also contained in the multiplicative closure, namely

$$(\mathrm{i}S_i)^2(-\mathrm{i}S_i) = \mathrm{i}S_i^3 = \mathrm{i}S_i^{-1} = (-\mathrm{i}S_i)^{-1}$$

Therefore from (3.2) one obtains

$$\mathrm{i}S_i^{-1}\exp(-\mathrm{i}\theta J_j)(-\mathrm{i}S_i) = \epsilon_{ijk}\exp(\mathrm{i}\theta H_k) + \delta_{ij}\exp(-\mathrm{i}\theta J_j)$$
$$\mathrm{i}S_i^{-1}\exp(-\mathrm{i}uK_j)(-\mathrm{i}S_i) = \epsilon_{ijk}\exp(-\mathrm{i}uI_k) + \delta_{ij}\exp(-\mathrm{i}uK_j).$$

Hence the extended relativity group in this case also contains all the elements of the complex Lorentz group $L_+(\mathrm{C})$. In particular, one reaches the conclusion that if equations (2.5) are chosen for the superluminal transformations then those of equations (2.3) are also contained in the group. (The converse is not true, of course.)

In particular, from

$$\mathrm{i}S_i^{-1}\exp(\mathrm{i}\tfrac{1}{2}\pi J_j)(-\mathrm{i}S_i) \doteq \epsilon_{ijk}S_k \qquad i \neq j,$$

one concludes that the superluminal transformations S_i are also contained in the new extended relativity group. Therefore, multiplying $\mathrm{i}S_i^{-1}$ by S_i one concludes that the closure is $\{L_+(\mathrm{C}), \mathrm{i}L_+(\mathrm{C})\}$. Therefore the minimal extensions of special relativity to include superluminal-like transformations lead either to $L_+(\mathrm{C})$ or to its central extension $\{L_+(\mathrm{C}), \mathrm{i}L_+(\mathrm{C})\}$.

The construction of unitary irreducible representations (UIR) is naturally our main tool for the exploration of extended relativity as a symmetry group. As far as the construction of UIR's is concerned, the consequences of the two extended relativity groups are the same because any UIR for $L_+(\mathrm{C})$ is also irreducible for $\{L_+(\mathrm{C}), \mathrm{i}L_+(\mathrm{C})\}$ and is obtained from the first by adjoining the i matrix, which is always unitary.

The main conclusion of this section is that no matter which reasonable way one chooses to extend special relativity by superluminal-like transformations, one is necessarily led to replace the Lorentz group by its complex counterpart. Therefore, despite the fact that physical observations take place (and are parametrised) in a real space–time, the exploration of extended relativity suggests that for theoretical purposes it might be convenient to consider this real space to be embedded in a complex four-dimensional manifold. This possibility leads to many interesting considerations and speculations, which, however, will not concern us in this paper. Here we will simply explore the possibility (and consequences) that some little group-irreducible representations of the real Poincaré group may also be irreducible for the complex group. We will call this small subset of representations of the complex group, somewhat arbitrarily, 'physically admissible representations'. The precise characterisation of this notion and its consequences will be the subject of the next section.

4. Physically admissible representations of extended relativity

Because extended relativity is so closely related to the complex Lorentz group, the representations to explore for possible physical consequences are obviously those of the associated complex Poincaré group $P_+(\mathrm{C})$ defined by

$$U[b, (A_1, B_1)]U[a, (A_2, B_2)] = U[b + \Lambda(A_1, B_1)a, (A_1A_2, B_1B_2)]$$

where a and b are complex four-vectors and the pairs of $\mathrm{SL}(2, \mathrm{C})$ matrices (A, B) defined the elements of the complex Lorentz group (see appendix 1). To construct the unitary irreducible representations of $P_+(\mathrm{C})$ one diagonalises the subgroup $\{U[a, (1, 1)]\}$, i.e. one chooses a basis of generalised momentum eigenvectors $|p\alpha\rangle$ such that

$$U[a, (1, 1)]|p\alpha\rangle = \exp(\mathrm{i}\,\mathrm{Re}(p\,.\,a))|p\alpha\rangle \tag{4.1}$$

where p and a are, in general, complex numbers. For a complex Lorentz transformation operating in this state

$$U[0, (A, B)]|p\alpha\rangle = \sum_\beta |\Lambda(A, B)p\beta\rangle D_{\beta\alpha}[(A, B); p] \tag{4.2}$$

and the representation matrix $D_{\beta\alpha}$ satisfies

$$D[(A_1, B_1); \Lambda(A_2, B_2)p]D[(A_2, B_2); p] = D[(A_1A_2, B_1B_2); p].$$

One should now proceed, in the usual way, to classify all possible little (isotropy) groups, choosing standard vectors in each orbit, etc. At this point, however, we restrict our study to a small subset of all possible representations of the complex Poincaré group, namely to those which are as close as possible to irreducible representations of the real Poincaré group.

The following two conditions define the restricted class of representations.

(1) The quantum number of $p^2 = p_\mu p^\mu$ is restricted to real values and the standard vectors in the orbits are also chosen to be real. Once p^2 and a standard vector are chosen, the irreducible representations of the complex isotropy group will in general contain a certain number of irreducible representations of the corresponding real isotropy subgroup.

(2) We will restrict ourselves to those irreducible representations of the complex isotropy group which are also irreducible for the real isotropy subgroup.

The physical significance of the class of representation defined by (1) and (2) will be discussed shortly. We now proceed to their determination.

(*a*) For $p^2 > 0$ a real standard vector has the typical form $(1, 0, 0, 0)$. From equations (A.8) and (A.9) in appendix 1, one concludes that the isotropy group for this standard vector is a SL(2, C) group generated by J_k and I_k. The corresponding isotropy group for the real Poincaré group is the rotation group generated by J_k.

Among the unitary irreducible representations of SL(2, C) (Naimark 1964), the only one that is irreducible for the rotation subgroup is the trivial one-dimensional identity representation (spin zero). All other UIR's are spin towers containing all integer or half-integer spins once. Hence, according to our specifications, the 'physically admissible' representations for the massive ($p^2 > 0$) case are only those of zero spin.

(*b*) For $p^2 = 0$ with $p^\mu \neq 0$, a standard vector satisfying the condition (1) is $(1, 0, 0, 1)$. From equation (A.1) a simple calculation shows that the isotropy group for this vector can be parametrised in terms of SL(2, C)$\times$SL(2, C) matrices as follows:

$$G[\rho, \alpha, a\,.\,b] = \left[\begin{pmatrix} \exp(\frac{1}{2}\rho + \frac{1}{2}\mathrm{i}\alpha) & a\exp(-\frac{1}{2}\rho - \frac{1}{2}\mathrm{i}\alpha) \\ 0 & \exp(-\frac{1}{2}\rho - \frac{1}{2}\mathrm{i}\alpha) \end{pmatrix}\begin{pmatrix} \exp(-\frac{1}{2}\rho + \frac{1}{2}\mathrm{i}\alpha) & b\exp(\frac{1}{2}\rho - \frac{1}{2}\mathrm{i}\alpha) \\ 0 & \exp(\frac{1}{2}\rho - \frac{1}{2}\mathrm{i}\alpha) \end{pmatrix}\right]$$

where ρ and α are real parameters and a and b are complex numbers.

The group law is

$$G[\rho_1, \alpha_1, a_1, b_1]G[\rho_2, \alpha_2, a_2, b_2] = G[\rho_1 + \rho_2, \alpha_1 + \alpha_2, a_1 + \exp(\rho_1 + \mathrm{i}\alpha_1)a_2, b_1 + \exp(-\rho_1 + \mathrm{i}\alpha_1)b_2].$$

The representations are obtained by diagonalising the subgroup $\{G[0, 0, a, b]\}$, i.e., by choosing a basis of vectors $|\pi, \ldots\rangle$ such that

$$G[0, 0, a, b]|\pi, \ldots\rangle = \exp(\mathrm{i}\boldsymbol{\pi}\cdot\boldsymbol{v})|\pi, \ldots\rangle$$

where $\boldsymbol{v} = (a, b)$ is a four-dimensional vector whose components are the real and imaginary parts of the parameters a and b.

As in the case of the real Poincaré group, there are continuous spin representations for $\boldsymbol{\pi} \neq 0$ and discrete spin ones for $\boldsymbol{\pi} = 0$. Restricting ourselves to this last class, which already in the real Poincaré group seems to be the one with physical relevance, one finds that in this case the irreducible representations of the isotropy group are one-dimensional and labelled by two real parameters ν and μ,

$$G[\rho, \alpha, a, b]|\boldsymbol{\pi} = 0\nu\mu\rangle = \exp[\mathrm{i}(\rho\nu + \alpha\mu)]|\boldsymbol{\pi} = 0\nu\mu\rangle \tag{4.3}$$

with the parameter μ taking the values $0, \pm\frac{1}{2}, \pm 1, \ldots$ and ν being an arbitrary real number.

The discrete spin isotropy group for the real Poincaré group corresponds to the subgroup $\{G[0, \alpha, a, a]\}$; therefore all the irreducible representations derived above are also irreducible for this subgroup. Hence all the discrete-spin massless representations of the complex Poincaré group are physically admissible according to our definition.

In this paper we are mainly interested in the consequences of extended relativity for massive and massless states. Therefore we will skip the study of the $p^2 < 0$ representations.

The main point in restricting our study to the small class of representations which we called 'physically admissible' is the following. Of course, we do not expect the extended (complex) relativity group to be a symmetry of general validity in nature. If it were, instead of isolated massive particles of a given spin we would have found in the physical spectrum spin towers of degenerate mass. Nevertheless, there is the possibility that, when invariance under the complex group does not require the introduction of more spins in each irreducible subspace, then some physical states might admit a space–time symmetry higher than the usual (real) Lorentz symmetry. This possibility seems particularly likely in the case of massless states where all discrete spin representations of the real group admit a simple extension to the complex group.

If there are states that, although having the same spin degrees of freedom as imposed by the real Lorentz group, possess as their kinematical symmetry group the larger complex group, this will have two main consequences.

(1) Their physical space of real momentum quantum numbers will have to be considered as embedded in a larger space that contains complex momenta.

(2) Because the complex Lorentz group $L_+(\mathrm{C})$ contains the operation PT as a transformation continuously connected to the identity, this operation will necessarily be represented in these spaces by an unitary operator.

In the remainder of this paper we will mainly explore the non-trivial implications of (2). In particular, we will assume that, while most physical states observed so far carry irreducible representations of the real Poincaré group with the discrete PT transformation being unconnected and anti-unitary, there might exist some other states that, having as a kinematical symmetry group the larger complex Poincaré group, are embedded in an irreducible representation of $P_+(\mathrm{C})$ and will necessarily have PT

implemented by a unitary operator. We call this new type of particle (quantum fields) 'chronons'†.

The superselection rule that operates between the space of chronons and spaces of non-chronons (i.e. usual particles), as well as the structure of their interactions, will be discussed in the next section. As a preparation for this discussion we list here some results concerning the transformation properties of chronon operators and quantum fields.

A spin-zero massive or massless chronon quantum field will have a momentum expansion

$$\Phi(x) \sim \int \mathrm{d}k\, \delta(k^2 - m^2) a(k) \exp(-\mathrm{i}\,\mathrm{Re}(k \cdot x)) \tag{4.4}$$

where the integral runs over real and complex four-momenta restricted only by the mass-shell condition $p^2 = m^2$.

Because we are concerned only with the effect of the chronon fields on physical real momenta spaces we may use in this expansion only the part that corresponds to real k. Therefore, using different symbols for the operators that correspond to the positive and negative energy parts, i.e. defining $a(-p) = \eta b^+(p)$ with $\eta^2 = 1$, one may consider the restricted field Φ_r:

$$\Phi_r(x) = (2\pi)^{-3/2} \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega_p}} (a(p) \exp(-\mathrm{i}p \cdot x) + b^+(p) \exp(\mathrm{i}p \cdot x)). \tag{4.5}$$

From the general expression for a spin-zero operator

$$U[b, \Lambda] a(p) U^{-1}[b, \Lambda] = \exp(-\mathrm{i}\,\mathrm{Re}(p \cdot b)) a(\Lambda p)$$

where $[b, \Lambda]$ is a general element of the complex Poincaré group it follows that

$$U[0, PT] a(p) U^{-1}[0, PT] = a(-p) = \eta b^+(p)$$

where in the last equality the redefinition of $a(-p)$ is used. Therefore, if one sticks to the usual convention of calling the a states particles and the b states antiparticles, one sees that for chronons the PT operation is unitary and transforms particle destruction operators into antiparticle creation operators and vice versa.

From now on we will label the operation PT, as well as T and P, in the space of chronons by the index c to distinguish them from the usual representations of these operations in the spaces of non-chronons:

$$(PT)_c a(p) (PT)_c^{-1} = \eta b^+(p). \tag{4.6}$$

For the quantum field $\Phi_r(x)$ the transformation law is

$$(PT)_c \Phi_r(x) (PT)_c^{-1} = \eta \Phi_r(-x), \tag{4.7}$$

in accordance with the physical interpretation of the PT operation.

The complex Lorentz group only specifies in a unique manner the PT transformation for chronons. For the calculations in the next section it is also convenient to have explicit forms for the separate P and T operations, and also for the charge-conjugation operation, C.

† Although the term 'chronon' has already been used by some authors in other contexts, I thought this would be an appropriate name for these new hypothetical states because of their presumed natural role in T violation.

Requiring the parity to have the usual properties, it follows that P_c, like $(PT)_c$, is a unitary operator. For the charge-conjugation operator C_c, if one wants to preserve *CPT* invariance for interactions of chronons with non-chronons (see next section), one is inevitably led to an anti-unitary operator.

In the table below we list the *C*, *P* and *T* transformations for the creation operators and quantum fields of both spinless chronons and non-chronons.

Chronons	Non-chronons
Complete inversion (*PT*)	
$(PT)_c a^+(k)(PT)_c^{-1} = a^+(-k) \equiv \eta b(k)$	$(PT)a^+(k)(PT)^{-1} = \eta a^+(k)$
unitary	anti-unitary
$(PT)_c\Phi(x)(PT)_c^{-1} = \eta\Phi(-x)$	$(PT)\Phi(x)(PT)^{-1} = \eta\Phi(-x)$
Space inversion	
$P_c a^+(k)P_c^{-1} = \eta_P a^+(k^0, -\boldsymbol{k})$	$Pa^+(k)P^{-1} = \eta_P a^+(k^0, -\boldsymbol{k})$
unitary	unitary
$P_c\Phi(x)P_c^{-1} = \eta_P\Phi(x^0, -\boldsymbol{x})$	$P\Phi(x)P^{-1} = \eta_P\Phi(x^0, -\boldsymbol{x})$
Time reversal	
$T_c a^+(k)T_c^{-1} = \eta_T b(k^0, -\boldsymbol{k})$	$Ta^+(k)T^{-1} = \eta_T a^+(k^0, -\boldsymbol{k})$
unitary	anti-unitary
$T_c\Phi(x)T_c^{-1} = \eta_T\Phi(-x^0, \boldsymbol{x})$	$T\Phi(x)T^{-1} = \eta_T\Phi(-x^0, \boldsymbol{x})$
Charge conjugation	
$C_c a^+(k)C_c^{-1} = \eta\eta_C b^+(-k) \equiv \eta_C a(k)$	$Ca^+(k)C^{-1} = \eta_C b^+(k)$
anti-unitary	unitary
$C_c\Phi(x)C_c^{-1} = \eta_C\Phi^+(x)$	$C\Phi(x)C^{-1} = \eta_C\Phi^+(x)$

For chronons of non-zero spin, which according to our specifications are necessarily massless, a slightly larger amount of work is required to obtain the transformations of the quantum fields. First a careful definition is needed for the elements of the complementary set which lead from the standard vector to general momentum states. A possible choice which constitutes a natural extension of the helicity complementary set in the real Poincaré group is the following.

Consider the definition of the massless state $|k\mu\rangle$ with k complex. Define as the 'direction of k' the three-space direction of Re k. The state $|k\mu\rangle$ can then be defined as

$$|k\mu\rangle = R_{kk^0} \exp(-\mathrm{i}\phi H_3) \exp(-\mathrm{i}uK_3)|k_s\mu\rangle \tag{4.8}$$

where $|k_s\mu\rangle$ is the standard vector, $\exp(-\mathrm{i}uK_3)$ 'boosts' the standard momentum $(k_s^0, 0, 0, k_s^0)$ to a momentum $|k|$ along the three-direction, $\exp(-\mathrm{i}\phi H_3)$ is a complex rotation that leads to a momentum Re k + i Im k along the three-direction and R_{kk^0} is a real rotation that rotates this vector to its final direction,

$$k^0 = k^3 = (\cos\phi - \mathrm{i}\sin\phi)|k|.$$

Notice that when applying the K_3 and H_3 transformations the eigenvalue of J_3 does not change and R_{kk^0} preserves the helicity. However, if Re $k < 0$ the appropriate H_3

rotation has to change the signs of the real parts of both the time and space components of the momentum; therefore μ will in this case represent the projection of J_3 along a direction opposite to Re $\boldsymbol{k}$. Therefore, for k real if $k^0>0$, μ is the helicity and if $k^0<0$, μ is minus the helicity.

It is now straightforward to obtain the action of the PT operation in an arbitrary $|k\mu\rangle$ state.

Defining

$$C(k, k_s) = \mathrm{R}_{kk^0} \exp(-\mathrm{i}\Phi H_3) \exp(-\mathrm{i}uK_3)$$

and restricting ourselves as before to k real,

$$U(PT)|k\mu\rangle = C(\underset{\sim}{P}Tk, k_s)C^{-1}(\underset{\sim}{P}Tk, k_s)\underset{\sim}{P}TC(k, k_s)|k_s\mu\rangle$$

and using $PT = \exp(-\mathrm{i}\pi H_3)\exp(-\mathrm{i}\pi J_3)$ one obtains

$$U(PT)|k\mu\rangle = \exp(-\mathrm{i}\pi\mu)|\underset{\sim}{P}Tk\mu\rangle. \tag{4.9}$$

For the creation operators the resulting unitary transformation becomes

$$U(PT)a^+(k\mu)U^{-1}(PT) = \exp(-\mathrm{i}\pi\mu)a^+(-k\mu).$$

Remembering that μ is the helicity for $k^0>0$ and $(-1)\times$helicity for $k^0<0$, one defines, in a manner similar to that for the spinless case, the 'antiparticle' operator

$$a^+(-k\mu) \equiv \eta \exp(\mathrm{i}\pi\mu)b(k-\mu)$$

and using, as before, the notation $(PT)_c$ for the representation of the PT automorphism in chronon space one obtains

$$(PT)_c a^+(k\mu)(PT)_c^{-1} = \eta b(k-\mu), \tag{4.10}$$

to be compared with the corresponding anti-unitary realisation in non-chronon spaces:

$$(PT)a^+(k\mu)(PT)^{-1} = \eta' a^+(k-\mu). \tag{4.11}$$

We are now ready to construct quantum fields for spinning massless chronons and find their transformation properties.

We list here the result for the spin-$\frac{1}{2}$ case which will be used in the examples of the next section. The free quantum field (real k part only) is defined as

$$\psi(x) = \sum_{\pm\mu}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}(a(k\mu)u'(k\mu)\exp(-\mathrm{i}kx) + b^+(k\mu)v'(k\mu)\exp(\mathrm{i}kx)) \tag{4.12}$$

where $u'(k\mu)$ and $v'(k\mu)$ are the following massless limits of helicity spinors (see appendix 2):

$$u'(k\mu) = \lim_{m\to 0}(m/k^0)^{1/2}u(k\mu)$$

$$v'(k\mu) = \lim_{m\to 0}(m/k^0)^{1/2}v(k\mu).$$

These, together with equation (4.12), guarantee that the quantum field has the usual Bjorken–Drell normalisation.

Considering, as in the spinless case, an unitary parity and anti-unitary charge-conjugation transformation for chronons and the properties of the spinors listed in appendix 2, the following table is obtained.

Chronons	Non-chronons
Complete inversion (PT)	
$(PT)_c a^+(k\mu)(PT)_c^{-1} = \exp(-i\pi\mu)a^+(-k\mu) \equiv \eta b(k-\mu)$	$(PT)a^+(k\mu)(PT)^{-1} = i\eta(-1)^{\epsilon(-\mu)}a^+(\boldsymbol{k}-\mu)$
$(PT)_c b^+(k\mu)(PT)_c^{-1} = \eta a(k-\mu)$	$PTb^+(k\mu)(PT)^{-1} = i\eta(-1)^{\epsilon(-\mu)}b^+(\boldsymbol{k}-\mu)$
unitary	anti-unitary
$(PT)_c\psi(x)(PT)_c^{-1} = \eta\gamma^5\psi(-x)$	$PT\psi(x)(PT)^{-1} = \eta\gamma^5\gamma^2\psi(-x)$
Space inversion	
$P_c a^+(k\mu)P_c^{-1} = \eta_P \exp(-i\theta(k,\mu)) \times a^+(-\boldsymbol{k}-\mu)$	$Pa^+(k\mu)P^{-1} = \eta_P \exp(-i\theta(k,\mu))a^+(-\boldsymbol{k}-\mu)$
$P_c b^+(k\mu)P_c^{-1} = -\eta_P \exp(i\theta(k,-\mu)) \times b^+(-\boldsymbol{k}-\mu)$	$Pb^+(k\mu)P^{-1} = -\eta_P \exp(i\theta(k,-\mu) \times b^+(-\boldsymbol{k}-\mu)$
unitary	unitary
$P_c\psi(x)P_c^{-1} = \eta_P\gamma^0\psi(x^0,-\boldsymbol{x})$	$P\psi(x)P^{-1} = \eta_P\gamma^0\psi(x^0,-\boldsymbol{x})$
Time reversal	
$T_c a^+(k\mu)T_c^{-1} = -\eta_T \exp(-i\theta(k\mu))b(-\boldsymbol{k}\mu)$	$Ta^+(k\mu)T^{-1} = i\eta_T(-1)^{\epsilon(-\mu)} \exp(-i\theta(k-\mu)) \times a^+(-\boldsymbol{k}\mu)$
$T_c b^+(k\mu)T_c^{-1} = \eta_T \exp(i\theta(k,-\mu))a(-\boldsymbol{k}\mu)$	$Tb^+(k\mu)T^{-1} = -i\eta_T(-1)^{\epsilon(-\mu)} \exp(-i\theta(k\mu)) \times b^+(-\boldsymbol{k}\mu)$
unitary	anti-unitary
$T_c\psi(x)T_c^{-1} = \eta_T\gamma^5\gamma^0\psi(-x_1^0\boldsymbol{x})$	$T\psi(x)T^{-1} = \eta_T i\gamma^1\gamma^3\psi(-x^0,\boldsymbol{x})$
Charge conjugation	
$C_c a^+(k\mu)C_c^{-1} = i\eta_C a(k\mu)$	$Ca^+(k\mu)C^{-1} = \eta_C b^+(k\mu)(-1)^{\epsilon(\mu)}$
$C_c b^+(k\mu)C_c^{-1} = -i\eta_C b(k\mu)$	$Cb^+(k\mu)C^{-1} = \eta_C a^+(k\mu)(-1)^{\epsilon(\mu)}$
anti-unitary	unitary
$C_c\psi_\alpha(x)C_c^{-1} = -i\eta_C\psi_\alpha^+(x)$	$C\psi_\alpha(x)C^{-1} = \eta_C(i\gamma^2)_{\alpha\beta}\psi_\beta^+(x)$

Here $\eta = \eta_P\eta_T$, the phases $\theta(k\mu)$ and $\epsilon(\mu)$ are

$$\theta(k+) = -\arg(k^1 + ik^2) \qquad \epsilon(+) = 0$$

$$\theta(k-) = -\arg(-k^1 + ik^2) \qquad \epsilon(-) = 1$$

and the phase in the charge-conjugation transformation of chronon fields was chosen so that chronons and non-chronons have the same CPT transformation properties, namely

$$(CPT)\psi_\alpha(x)(CPT)^{-1} = -i\eta\eta_C(\gamma^5)_{\alpha\beta}\psi_\beta^+(-x).$$

From the table the transformation properties for any spinor bilinears follows in a straightforward manner. For the purpose of the applications in the next section we list the following.

Chronons	Non-chronons
$\bar{\psi}\psi(x) \xrightarrow{P_c} \bar{\psi}\psi(x^0, -\boldsymbol{x})$	$\bar{\psi}\psi(x) \xrightarrow{P} \bar{\psi}\psi(x^0, -\boldsymbol{x})$
$\bar{\psi}\psi(x) \xrightarrow{T_c} -\bar{\psi}\psi(-x^0, \boldsymbol{x})$	$\bar{\psi}\psi(x) \xrightarrow{T} \bar{\psi}\psi(-x^0, \boldsymbol{x})$
$[\bar{\psi}, \psi](x) \xrightarrow{C_c} -[\bar{\psi}, \psi](x)$	$[\bar{\psi}, \psi](x) \xrightarrow{C} [\bar{\psi}, \psi](x)$
$\bar{\psi}\gamma^5\psi(x) \xrightarrow{P_c} -\bar{\psi}\gamma^5\psi(x^0, -\boldsymbol{x})$	$\bar{\psi}\gamma^5\psi(x) \xrightarrow{P} -\bar{\psi}\gamma^5\psi(x^0, -\boldsymbol{x})$
$\bar{\psi}\gamma^5\psi(x) \xrightarrow{T_c} \bar{\psi}\gamma^5\psi(-x^0, \boldsymbol{x})$	$\bar{\psi}\gamma^5\psi(x) \xrightarrow{T} \bar{\psi}\gamma^5\psi(-x^0, \boldsymbol{x})$
$[\bar{\psi}, \gamma^5\psi](x) \xrightarrow{C_c} [\bar{\psi}, \gamma^5\psi](x)$	$[\bar{\psi}, \gamma^5\psi](x) \xrightarrow{C} [\bar{\psi}, \gamma^5\psi](x)$

5. Chronons and symmetry breaking

A first consequence of the PT transformation properties of chronons is the result that there is a superselection rule operating between the spaces of chronons and non-chronons.

Let V_c denote the space of free chronons and V the space of ordinary matter. Consider a vector $|\psi\rangle = \lambda|c\rangle + \mu|\alpha\rangle$ in $V_c \oplus V$ where $|c\rangle \in V_c$, $|\alpha\rangle \in V$ and λ, μ are real numbers. The vectors $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ belong to the same ray, and should therefore represent the same physical space. Let us apply the PT transformation to these vectors. For $|\psi\rangle$ we obtain

$$\underset{\sim}{PT}|\psi\rangle = \lambda(PT)_c|c\rangle + \mu(PT)|\alpha\rangle$$

and for $e^{i\theta}|\psi\rangle$

$$\underset{\sim}{PT}(e^{i\theta}|\psi\rangle = \lambda\, e^{i\theta}(PT)_c|c\rangle + \mu\, e^{-i\theta}(PT)|\alpha\rangle.$$

Therefore $\underset{\sim}{PT}|\psi\rangle$ and $\underset{\sim}{PT}(e^{i\theta}|\psi\rangle)$ belong to different rays; hence $\underset{\sim}{PT}$ does not establish a ray correspondence in $V_c \oplus V$ unless $\lambda = 0$ or $\mu = 0$, i.e. unless chronons and ordinary matter belong to distinct superselection sectors.

To say that free chronons and ordinary matter belong to different superselection sectors has as yet no particular consequences on their mutual interactions, because the above result provides information only on the structuture of the direct sum space $V_c \oplus V$. The nature of the interactions is actually related to the structure of the tensor space $V_c \otimes V$.

Denote by $|c_i\alpha_i\rangle = |c_i\rangle \otimes |\alpha_i\rangle$ with $|c_i\rangle \in V_c$, $|\alpha_i\rangle \in V$ an arbitrary basis state in $V_c \otimes V$. Let $\mathbf{U}$ and $\mathbf{A}$ be the operators that implement the PT transformation in the spaces V_c and V respectively, i.e. $\mathbf{U} \equiv (PT)_c$ and $\mathbf{A} \equiv (PT)$.

To study the nature of the tensor product operator $\mathbf{U} \otimes \mathbf{A}$ in $V_c \otimes V$, consider the matrix element

$$\begin{aligned}(\mathbf{U}\otimes\mathbf{A}c_2\alpha_2, \mathbf{U}\otimes\mathbf{A}c_1\alpha_1) &= (c_2, c_1)(\alpha_1, \alpha_2)\\ &= (c_2\alpha_1, c_1\alpha_2).\end{aligned}$$

Therefore there is no choice of phase that can make $\mathbf{U} \otimes \mathbf{A}$ unitary or anti-unitary in the tensor product space. According to Wigner's theorem, the full PT transformation cannot be implemented as a symmetry in the (scattering) tensor space.

To discuss the implications of this result one should recall some facts about the role of symmetry operations in physical spaces.

A symmetry of a theory, i.e. an automorphism τ of its algebra of observables A, is said to be conserved in a particular representation space if it can be implemented in this space by unitary or anti-unitary operators and if it commutes with time evolution (or at least it has well defined commutation properties, as in the case of Lorentz transformations).

This second condition is necessary because otherwise the automorphism defined as a transformation on the operators at a given time t would not be consistent with the transformation at some other time. This second requirement is expressed in (Lagrangian) field theory by requiring the equations of motion (or the Lagrangian) to be invariant under the transformation.

In general, when a symmetry is broken it is because this second requirement fails. An exception is the case of the so called 'spontaneous breaking of symmetry' (SBS) where the equations of motion (and the Lagrangian) are invariant, but the automorphism cannot be implemented in the physical space.

In our case the PT automorphism cannot be implemented in the tensor space, at least in a way that is consistent with the properties of the operators that implement it in the free particle spaces; i.e. we face a situation where 'spontaneous breaking' can occur.

Three important remarks should be made at this point.

(*a*) In the known cases of spontaneous symmetry breaking what usually occurs is that the operator that would implement the automorphism when applied to vectors in the irreducible representation space leads to vectors outside the space. Therefore it would be possible, in principle, to construct an operator representation of the automorphism if the space of states were enlarged to a reducible direct sum (integral) over irreducible spaces. That is, the automorphism which is not implementable in a single irreducible vector space becomes implementable in the fibre bundle over the set of degenerate ground states.

In the present case, however, no such extension can make the automorphism implementable and we could say that we have found a new type of spontaneous symmetry breaking.

(*b*) Chronons, as we have defined them, possess a kinematical invariance group (the complex Poincaré group) larger than the one of ordinary matter. Therefore if they are to interact at all with ordinary matter these interactions should not spoil this larger invariance.

In particular, $(PT)_c$ is an element of the kinematical group, and in the Lagrangian the terms corresponding to interactions of chronons with ordinary matter should always be invariant under the action of an *unitary* $(PT)_c$ operator, i.e., there should be a unitary automorphism operating in the full $V_c \otimes V$ (which for the V_c part is interpreted as the realisation of the physical PT transformation but for the V part has no special physical meaning). The consequence is that there can be no actual violation of PT in chronon space and what this special type of spontaneous symmetry breaking can do at most is to induce PT violation in the space of ordinary matter. This situation will become clear in the examples discussed later.

(*c*) Non-implementability of an automorphism is the general symptom of spontaneous symmetry breaking, but how does one explore it in practice?

Because it is simpler to explore the dynamics from a Lagrangian than from the field algebra directly, what one usually does is to choose a representation for the fields appropriate to a particular vacuum among the set of degenerate ones and to rewrite the explicitly invariant Lagrangian in terms of shifted fields. When written in terms of the shifted fields, the Lagrangian is no longer explicitly invariant and the calculations may proceed as if the symmetry breaking were manifest rather than of the spontaneous type. In our examples below, this is also to be the method that we use, and because the character of a PT transformation is a phase our shifted fields are 'phase-shifted fields'.

The interaction term to be explored in the examples is

$$L_{\mathrm{I}} = g[\bar{\psi}, \Gamma\psi](\phi + \phi^{+}) \tag{5.1}$$

where Γ is either 1 or $\mathrm{i}\gamma^5$. By choosing the signs of η_P, η_T and η_C such a term can be made invariant for both P, C, T (T anti-unitary) and P_{c}, C_{c}, T_{c} (C_{c} anti-unitary) (see the transformation tables in the previous section).

As far as P_{c}, C_{c} and T_{c} are concerned, L_{I} is one of a family of equally invariant Hermitian Lagrangians $\{L_{\mathrm{I}}^{\alpha}\}$:

$$L_{\mathrm{I}}^{\alpha} = g[\bar{\psi}, \Gamma\psi](\phi\, \mathrm{e}^{\mathrm{i}\alpha} + \phi^{+}\, \mathrm{e}^{-\mathrm{i}\alpha}). \tag{5.2}$$

Although invariant for P_{c}, C_{c} and T_{c} (if $L_{\mathrm{I}} \equiv L_{\mathrm{I}}^{\alpha}$ is invariant), for $\alpha \neq 0$ these interaction terms are not invariant under C or T. That is,

$$PL_{\mathrm{I}}^{\alpha}(x)P^{-1} = L_{\mathrm{I}}^{\alpha}(x^0, -\boldsymbol{x})$$
$$(CT)L_{\mathrm{I}}^{\alpha}(x)(CT)^{-1} = L_{\mathrm{I}}^{\alpha}(-x^0, \boldsymbol{x})$$

but

$$CL_{\mathrm{I}}^{\alpha}(x)C^{-1} = [\bar{\psi}, \Gamma\psi](\phi^{+}\, \mathrm{e}^{\mathrm{i}\alpha} + \phi\, \mathrm{e}^{-\mathrm{i}\alpha}) \neq L_{\mathrm{I}}^{\alpha}(x)$$
$$TL_{\mathrm{I}}^{\alpha}(x)T^{-1} = [\bar{\psi}, \Gamma\psi](-x^0, \boldsymbol{x})(\phi(-x^0, \boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\alpha} + \phi^{+}(-x^0, \boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\alpha}) \neq L_{\mathrm{I}}^{\alpha}(-x^0, \boldsymbol{x}).$$

P and CT are good symmetries for any $\alpha \neq 0$ but C and T are not separately conserved.

The replacements $\phi \to \mathrm{e}^{\mathrm{i}\alpha}\phi$ and $\phi^{+} \to \mathrm{e}^{-\mathrm{i}\alpha}\phi$ that occur on passing from L_{I} to L_{I}^{α} correspond precisely to the choice of a representation where C (and T) is not implemented, and this is, as we have discussed above, the standard way to explore practically the consequences of spontaneous symmetry breaking (SSB). That is, once we know that the automorphism cannot be implemented in the scattering space one shifts the fields by fixing for them a particular representation where the transformation is not implemented, and thus one transfers to an equivalent explicitly symmetry breaking Lagrangian the effects of SSB. In fact, the shifts $\phi \to \mathrm{e}^{\mathrm{i}\alpha}\phi$ and $\phi^{+} \to \mathrm{e}^{-\mathrm{i}\alpha}\phi^{+}$ correspond to the replacements

$$a^{+}(k) \to a^{+}(k)' = \mathrm{e}^{-\mathrm{i}\alpha}a^{+}(k)$$
$$b^{+}(k) \to b^{+}(k)' = \mathrm{e}^{\mathrm{i}\alpha}b^{+}(k).$$

Therefore if $Ca^{+}(k)C^{-1} = \eta_C b^{+}(k)$ and $Ta^{+}(k)T^{-1} = \eta_T a^{+}(k^0, -\boldsymbol{k})$ the correspondence is no longer preserved for the primed operators

$$Ca^{+}(k)'C^{-1} = \eta_C\, \mathrm{e}^{-\mathrm{i}\alpha}b^{+}(k) \neq \eta_C b^{+}(k)'$$
$$Ta^{+}(k)'T^{-1} = \eta_T\, \mathrm{e}^{\mathrm{i}\alpha}a^{+}(k^0, -\boldsymbol{k}) \neq \eta_T a^{+}(k^0, -\boldsymbol{k})'$$

whereas for C_c and T_c the appropriate correspondences are preserved under the shift

$$C_c a^+(k)' C_c = \eta_C \, e^{i\alpha} a(k) = \eta_C a(k)'$$
$$T_c a^+(k)' T_c = \eta_T \, e^{-i\alpha} b(k^0, -\boldsymbol{k}) = \eta_T b(k^0, -\boldsymbol{k})'.$$

In conclusion, the interaction of equation (5.2) provides us with a shifted Lagrangian appropriate to explore the practical consequences of this special type of SSB, which in addition satisfies the requirement of invariance under $(PT)_c$ that, as we saw, is a necessary condition to preserve the large kinematical invariance group of chronons.

Actually, if 'extended relativity' is a good kinematical symmetry for chronons, all that its interacting Lagrangian terms are required to satisfy is to be invariant under $(PT)_c$ (because it belongs to the kinematical group) and not under PT, which is not even unitary. Therefore we will always be led to physical processes where, from the point of view of the genuine Lagrangian symmetries, nothing is really broken.

Putting it simply, from the point of view of chronons any T-violating effects generated by their agency are perfectly symmetric processes. It is only our prejudice in extending to interactions of chronons with ordinary particles the same discrete symmetry conventions that one uses for the latter that leads to effects that, adhering to the old conventions, one describes as symmetry breaking.

The final part of this paper will be devoted to a study of shifted interaction terms of the form of equation (5.2) for two simple situations: first assuming that it represents an interaction between spin-$\frac{1}{2}$ massless chronons with scalar non-chronons, and secondly in the reserved situation where ϕ represents scalar massive or massless chronons and ψ is a spinor ordinary matter field.

5.1. *Example 1*

Let ψ be the field of a spinor (massless) chronon and ϕ a scalar ordinary matter field. The quantum numbers of the field are chosen to allow the study of the possible relevance of this model to T violation in the neutral kaon complex. Therefore

$$\begin{aligned}
&L_1^\alpha = ig\bar{\psi}\gamma^5\psi(\phi \, e^{i\alpha} + \phi^+ e^{-i\alpha}) + L_{ct}\\
&P\phi(x)P^{-1} = -\phi(x^0, -\boldsymbol{x}) = P_c\phi(x)P_c^{-1}\\
&C\phi(x)C^{-1} = \phi^+(x) = C_c\phi(x)C_c^{-1}\\
&T\phi(x)T^{-1} = -\phi(-x^0, \boldsymbol{x}) = T_c\phi(x)T_c^{-1}.
\end{aligned} \tag{5.3}$$

L_{ct} stands for the counter-term contribution needed for renormalisation.

Defining particle $a(k)$ and antiparticle $\bar{a}(k)$ operators from the expansion

$$\phi(x) = \int \frac{d^3k}{[(2\pi)^3 2\omega_k]^{1/2}} (a(k) \, e^{-ikx} + \bar{a}^+(k) \, e^{ikx}) \tag{5.4}$$

the Feynman rules for vertices in this theory are as shown in figure 1. To convert to familiar looking notation set

$$a^+(k)|0\rangle \equiv |K^0\rangle$$
$$\bar{a}^+(k)|0\rangle \equiv -|\bar{K}^0\rangle.$$

Figure 1. Vertex factors for model 1.

An arbitrary state

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \equiv \xi_1|K^0\rangle + \xi_2|\bar{K}\rangle^0\rangle$$

will evolve in time according to the equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \Lambda\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

where $\Lambda = M - \mathrm{i}\Gamma$ is the total mass matrix in the $\mathrm{K}^0-\bar{\mathrm{K}}^0$ representation and can be separated conventionally into Hermitian and anti-Hermitian parts ($M = M^+$ and $\Gamma = \Gamma^+$).

The eigenstates $|\mathrm{K}_S\rangle$ and $|\mathrm{K}_L\rangle$ of the mass matrix will evolve in time as

$$|\mathrm{K}_{S,L}\rangle_t = \exp(-\mathrm{i}\lambda_{S,L}t)|\mathrm{K}_{S,L}\rangle_0$$

with

$$\lambda_{S,L} = m_{S,L} - \tfrac{1}{2}\mathrm{i}\gamma_{S,L}.$$

These states are related to the previous ones by

$$|\mathrm{K}_S\rangle = \frac{1}{[2(1+|\epsilon|^2)]^{1/2}}[(1+\epsilon)|\mathrm{K}^0\rangle + (1-\epsilon)|\bar{\mathrm{K}}^0\rangle]$$

$$|\mathrm{K}_L\rangle = \frac{1}{[2(1+|\epsilon|^2)]^{1/2}}[(1+\epsilon)|\mathrm{K}^0\rangle - (1-\epsilon)|\bar{\mathrm{K}}^0\rangle]$$

where *CPT* invariance is already assumed.

The experimental parameter ϵ is related to the elements of the mass matrix as follows:

$$\mathrm{Im}\,\Gamma_{12} + \mathrm{i}\,\mathrm{Im}\,M_{12} = \frac{\epsilon}{1-\epsilon^2}(\lambda_S - \lambda_L) \simeq \epsilon(\lambda_S - \lambda_L) \tag{5.5}$$

where the last equality holds for small ϵ.

To $\mathrm{Im}\,M_{12}$ contribute, in this model, the $\mathrm{a}\to\bar{\mathrm{a}}$ virtual transitions, of which the lowest-order ones are drawn in figure 2(a), while to $\mathrm{Im}\,\Gamma_{12}$ contribute in lowest order the real $\mathrm{a}\to\mathrm{c}\bar{\mathrm{c}}$ and $\bar{\mathrm{a}}\to\mathrm{c}\bar{\mathrm{c}}$ transitions (figure 2(b)). These contributions will both be proportional to $g^2\sin 2\alpha$:

$$\mathrm{Im}\,\Gamma_{12} = \tfrac{1}{2}\,\mathrm{Im}\sum_n \langle n|T|\bar{\mathrm{K}}^0\rangle\langle n|T|\mathrm{K}^0\rangle^*. \tag{5.6}$$

Let us now compute the contributions of the graphs of figure 2:

$$\mathrm{Im}\,M_{12} \simeq \mathrm{Im}\left(\frac{\mathrm{i}}{2m_{\mathrm{K}}}\int \mathrm{Tr}\left[\frac{\mathrm{i}}{(\not{p}+\not{K})}\gamma^5\frac{\mathrm{i}}{\not{p}}\gamma^5\right]\frac{\mathrm{d}^4p}{(2\pi)^4}g^2\,\mathrm{e}^{-\mathrm{i}2\alpha}\right).$$

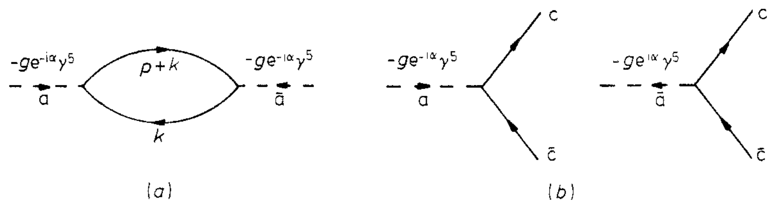

Figure 2. Lowest-order diagrams contributing to (*a*) Im M_{12} and (*b*) Im Γ_{12} in model 1.

Notice that the minus sign of the closed fermion loop of figure 2(*a*) was cancelled by the minus sign in our definition $|\bar{\mathrm{K}}^0\rangle \equiv -\bar{a}^+(k)|0\rangle$. Computing the trace,

$$\operatorname{Im} M_{12} \simeq \operatorname{Im} \frac{-\mathrm{i}g^2\,\mathrm{e}^{-\mathrm{i}2\alpha}}{m_{\mathrm{K}}}\left(2\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2} - k^2\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2(p+k)^2}\right). \tag{5.7}$$

To extract physical results from these badly divergent integrals is particularly delicate because the massless nature of the spinor chronon field forces us to cope with genuine infrared divergences. In particular, as pointed out by several authors (Leibbrandt 1975 and references therein), the convenient technique of dimensional regularisation requires some modification to deal with massless fields. The worst ambiguity comes from the first integral in equation (5.7), which appears frequently in quantum gravity calculations where it is called the 'tadpole integral'. To deal with these integrals, Leibbrandt and Capper (1974), for example, have proposed a redefinition of the generalised gaussian integral in n-dimensional Euclidean space which involves the addition of a 'continuity function' to the argument of the exponentials. What this function does, essentially, is to avoid the disappearance of the exponentials in the parametric integrals so that we can still have recourse to Euler's representation of the γ function.

Using the method of Leibbrandt and Capper, the 'tadpole integral' is regularised to zero and the second integral, after removing the poles at dimension four, yields

$$-\mathrm{i}\frac{m_{\mathrm{K}}^2}{(4\pi)^2}\frac{\psi(1)\Gamma^2(1)}{\Gamma(2)}.$$

However, as Capper and Leibbrandt (1974) themselves point out, one could choose other continuity functions and regularise the tadpole integral to any other finite value. Therefore we write our matrix element as

$$\operatorname{Im} M_{12} \simeq \frac{m_{\mathrm{K}}}{(4\pi)^2}\frac{\psi(1)\Gamma^2(1)}{\Gamma(2)}g^2\sin 2\alpha + I \tag{5.8}$$

where I is zero in the Capper and Leibbrandt method but it is left out explicitly in equation (5.8) to emphasise the basic ambiguity of this result.

For Im Γ_{12} the calculation proceeds from equation (5.6) without ambiguity and the result is

$$\operatorname{Im} \Gamma_{12} \simeq -\frac{m_{\mathrm{K}}}{16\pi}g^2\sin 2\alpha. \tag{5.9}$$

Because the spinor chronons are massless there will always be non-vanishing transition probabilities for the real decays a → cc̄ and ā → cc̄. Consequently, the model of this example is intrinsically a milliweak model for T violation.

Could it nevertheless be consistent with the experimental data when applied to the neutral kaon system? If $I=0$ the model is ruled out by experiment in the neutral kaon system because then, from equations (5.5), (5.8), (5.9) and $(\lambda_S-\lambda_L)_{\exp}\simeq(-0{\cdot}535-\mathrm{i}0{\cdot}559)\times10^{10}\,\mathrm{s}^{-1}$ one obtains

$$\epsilon\simeq(\lambda_S-\lambda_L)^{-1}g^2\sin2\alpha\frac{m_{\mathrm{K}}}{16\pi}\left(-1+\mathrm{i}\frac{\psi(1)}{\pi}\right)\simeq1{\cdot}3g^2\sin2\alpha\frac{m_{\mathrm{K}}}{16\pi}\exp(\mathrm{i}\theta) \tag{5.10}$$

with $\theta\simeq-35{\cdot}8°$.

The parameter-free prediction of the phase $\theta\simeq-35{\cdot}8°$ compares poorly with the experimental value

$$\epsilon\simeq\tfrac{1}{3}(2\eta_{+-}+\eta_{00})\simeq2{\cdot}29\times10^{-3}\exp(\mathrm{i}45°).$$

However, because of the ambiguity in the calculation of Im M_{12}, we cannot have full confidence in the result of equation (5.10). Therefore, leaving I undetermined and turning our reasoning around, we might say that because the data are still consistent with $|\mathrm{Im}\,\Gamma_{12}|\leqslant5\times10^5\,\mathrm{s}^{-1}$ it follows from equation (5.9) that $g^2\sin2\alpha$ need not be smaller than

$$g^2\sin2\alpha\leqslant0{\cdot}3\times10^{-16}.$$

Then, because $g^2\geqslant g^2\sin2\alpha$, this would imply that one could have decay rates for $\mathrm{K}^0,\bar{\mathrm{K}}^0\to\mathrm{c}\bar{\mathrm{c}}$ as large as $10^6\,\mathrm{s}^{-1}$. Such a rate is still comparable with the decay rates for $\mathrm{K}_L\to3\pi$; hence, if one could account for all kaons in a K_L decay experiment, one might find some 'missing kaons' corresponding to those that decay to chronon–antichronon pairs.

In conclusion, either the model of example 1 is inconsistent with experiment or, if consistent because it is a milliweak model, one might expect to find the 'missing kaon effect' mentioned above.

In the simple-minded comparisons of this model with the neutral kaon case the neutral field was identified with the kaon field. From what we seem to understand about compositeness, quark models, etc, it may not be such a good idea to consider kaons as elementary fields. However, this does not preclude the application of this simple model because it may be simply extended to a scheme where kaons are composite and which leads essentially to the same results. In that case $\phi(x)$ would be identified with an intermediate neutral boson field which interacts with chronons in the manner specified by L_1^α, and in a C, P, T-invariant manner with the quark fields. The relevant T-violating diagrams would then be those of figure 3, which clearly have the same consequences as those of figure 2.

5.2. Example 2

In the second example we consider a spinless Hermitian chronon field coupling to the strangeness-changing scalar neutral current $\bar{\mathrm{q}}\frac{1}{2}(\lambda_6-\mathrm{i}\lambda_7)\mathrm{q}$:

$$\begin{aligned}L_2^\alpha&=g\,\mathrm{e}^{\mathrm{i}\alpha}\bar{q}\tfrac{1}{2}(\lambda_6-\mathrm{i}\lambda_7)\mathrm{q}\phi+\mathrm{cc}+L_{\mathrm{ct}}\\&=g\,\mathrm{e}^{\mathrm{i}\alpha}\bar{s}\,\mathrm{d}\phi+g\,\mathrm{e}^{-\mathrm{i}\alpha}\,\bar{\mathrm{d}}s\,\phi+L_{\mathrm{ct}}\end{aligned}$$

where q is assumed to be a triplet of quark-like fields and the up, down and strange-quark notation is used in the second equality. The Feynman rules for the vertices are as displayed in figure 4.

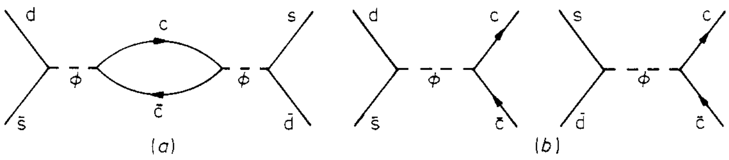

Figure 3. Diagrams for a modified model 1 with an intermediate boson field and composite kaons.

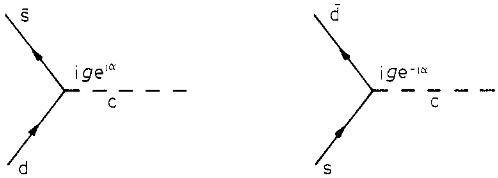

Figure 4. Vertex factors for model 2.

The lowest-order diagrams that contribute to Im M_{12} in this case are drawn in figure 5. The diagram of figure 5(b) is proportional to $g^2 \exp(\mathrm{i}2\alpha)/(m_K^2 - m_c^2)$. For the diagram of figure 5(a), if one considers the kaon as formed by quasifree quarks, as is implied by some confinement models, for the virtual $K^0 - \bar{K}^0$ transition it will be $p_1 \approx p_2$, $p_3 \approx p_4$ and $p_1 - p_3 \approx 0$. Therefore the contribution is proportional to $-g^2 \exp(\mathrm{i}2\alpha)/m_c^2$.

For Im Γ_{12} the lowest-order graphs that will contribute are drawn in figure 6. These graphs, which correspond to real $K^0 \to 3c$ and $\bar{K}^0 \to 3c$ transitions, are of higher order than those contributing to Im M_{12} and, furthermore, they will be non-vanishing only if $m_{K^0} > 3m_c$. Therefore the present model leads to an approximate hyperweak theory or even to an exact hyperweak theory if $m_{K^0} < 3m_c$, which is quite possible because spinless chronons may be massive.

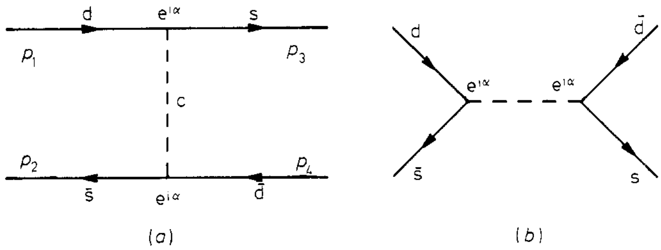

Figure 5. Lowest-order diagrams contributing to Im M_{12} in model 2.

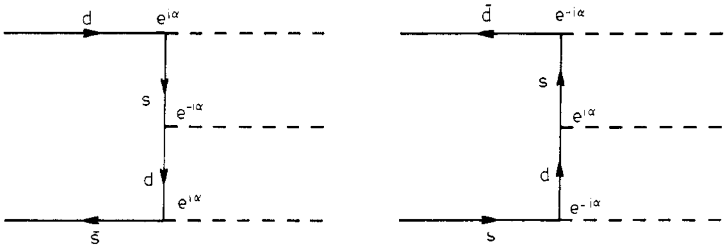

Figure 6. Lowest-order diagrams contributing to Im Γ_{12} in model 2.

Besides its possible relevance to T violation in the neutral kaon complex, the model also predicts some other processes. For example, in figure 7 we have drawn diagrams contributing to $\pi^0 \to 2c$ and $\eta \to 2c$ decays which, of course, will contribute to real processes only if $m_{\pi^0} > 2m_c$ and $m_\eta > 2m_c$ respectively. These processes are of second order in g, i.e. the same order as T violation in neutral kaons, but because of the opposite phases in the couplings at the two vertices they are T conserving.

If chronons are massive the best chance to look for them as real states would be in reactions. A potentially interesting one, implied by the present model, would be chronon production in kaon–nucleon collision $Kp \to cp$ (see figure 8), or the reverse reaction $cp \to Kp$ which might provide a convenient chronon detection method by looking for 'unaccompanied production of kaons'.

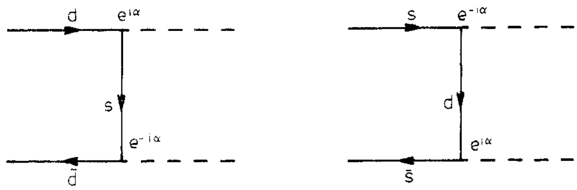

Figure 7. Diagrams contributing to π^0 and η decay into chronons.

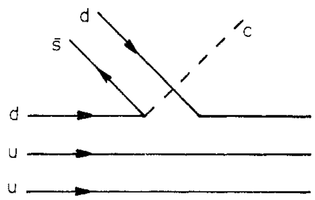

Figure 8. Chronon production in kaon–nucleon interactions.

In conclusion, it is quite possible that there might exist particles whose continuous kinematical invariance group is larger than the usual Poincaré group. If they do exist, although being otherwise non-exotic (i.e. non-superluminal), the very nature of their kinematical group has severe implications on their behaviour in relation to ordinary matter, namely there is a superselection rule and time reversal may be spontaneously broken. This last fact suggests that these states might be associated with the phenomenon of T violation in neutral kaon decays. Constructing models for this phenomenon, of which we gave two simple examples in this last section, one is able to predict processes where such particles might be found as real states and where it might be worthwhile to look for them. If they are found, their existence would have far-reaching implications concerning an underlying complex structure of the space–time manifold.

Appendix 1. Complex Lorentz group

The complex Lorentz group (of first kind) $L(\mathrm{C})$ is the group of 4×4 complex matrices Λ which preserves the metric matrix G with diagonal elements $(1, -1, -1, -1)$, i.e.

$$\Lambda^{\mathrm{T}} G \Lambda = G$$

i.e. $(\Lambda^k{}_\mu \Lambda_{k\nu} = g_{\mu\nu})$. The group has two connected components, $L_+(\mathrm{C})$ and $L_-(\mathrm{C})$, distinguished by the sign of the determinant, $\det \Lambda = \pm 1$.

There is a two-to-one homomorphism between SL(2, C) × SL(2, C) (i.e. the group of ordered pairs of SL(2, C) matrices) and $L_+(\mathrm{C})$. The correspondence is obtained by defining the coordinate matrix $\mathbf{x} = x^0 1 + \boldsymbol{x} \cdot \boldsymbol{\tau}$ and setting

$$\mathbf{x}' = A\mathbf{x}B^+. \tag{A.1}$$

Since $x'^\mu = \frac{1}{2}\,\mathrm{Tr}(\mathbf{x}'\tau^\mu)$ the complex Lorentz matrix corresponding to a given pair (A, B) is

$$\Lambda(A, B)^\mu{}_\nu = \tfrac{1}{2}\,\mathrm{Tr}(\tau^\mu A \tau^\nu B^+). \tag{A.2}$$

The *real* restricted Lorentz group corresponds to the diagonal part of SL(2, C) × SL(2, C), i.e. to the set of pairs (A, A).

The group has 12 generators. Denoting by $M^{\mu\nu}_{(1)}$ and $M^{\mu\nu}_{(2)}$ respectively the generators associated with the first and second SL(2, C) in SL(2, C) × SL(2, C)

$$[M^{\mu\nu}_{(1)}, M^{\tau\delta}_{(2)}] = 0 \tag{A.3}$$

and defining

$$\begin{aligned} M^{\mu\nu} &= M^{\mu\nu}_{(1)} + M^{\mu\nu}_{(2)} \\ N^{\mu\nu} &= M^{\mu\nu}_{(1)} - M^{\mu\nu}_{(2)} \end{aligned} \tag{A.4}$$

one obtains the following commutation relations:

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\tau}] &= \mathrm{i}(M^{\mu\rho}g^{\nu\tau} + M^{\nu\tau}g^{\mu\rho} - M^{\nu\rho}g^{\mu\tau} - M^{\mu\tau}g^{\nu\rho}) \\ [M^{\mu\nu}, N^{\rho\tau}] &= \mathrm{i}(N^{\mu\rho}g^{\nu\tau} + N^{\nu\tau}g^{\mu\rho} - N^{\nu\rho}g^{\mu\tau} - N^{\mu\tau}g^{\nu\rho}) \\ [N^{\mu\nu}, N^{\rho\tau}] &= \mathrm{i}(M^{\mu\rho}g^{\nu\tau} + M^{\nu\tau}g^{\mu\rho} - M^{\nu\rho}g^{\mu\tau} - M^{\mu\tau}g^{\nu\rho}). \end{aligned} \tag{A.5}$$

The $M^{\mu\nu}$'s are the generators of the real Lorentz group. Defining

$$\begin{aligned} J_i^{(\ )} &= -\tfrac{1}{2}\epsilon_{imn}M^{mn}_{(\ )} \\ K_i^{(\ )} &= M^{i0}_{(\ )} \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} J_m &= J_m^{(1)} + J_m^{(2)} \qquad & H_m &= J_m^{(1)} - J_m^{(2)} \\ K_m &= K_m^{(1)} + K_m^{(2)} \qquad & I_m &= K_m^{(1)} - K_m^{(2)} \end{aligned} \tag{A.7}$$

the commutation relations expressed in terms of this set of generators are

$$\begin{aligned} [J_m, J_n] &= \mathrm{i}\epsilon_{mnk}J_k \qquad & [H_m, H_n] &= \mathrm{i}\epsilon_{mnk}J_k \\ [J_m, K_n] &= \mathrm{i}\epsilon_{mnk}K_k \qquad & [H_m, I_n] &= \mathrm{i}\epsilon_{mnk}K_k \\ [K_m, K_n] &= -\mathrm{i}\epsilon_{mnk}J_k \qquad & [I_m, I_n] &= -\mathrm{i}\epsilon_{mnk}J_k \\ [J_m, H_n] &= \mathrm{i}\epsilon_{mnk}H_k \qquad & [J_m, I_n] &= \mathrm{i}\epsilon_{mnk}I_k \\ [K_m, H_n] &= \mathrm{i}\epsilon_{mnk}I_k \qquad & [K_m, I_n] &= -\mathrm{i}\epsilon_{mnk}H_k. \end{aligned} \tag{A.8}$$

Examples of some $L_+(\mathrm{C})$ coordinate transformations are

$$\exp(-\mathrm{i}uK_3)\begin{cases} x'^0 = \mathrm{ch}\, u\, x^0 + \mathrm{sh}\, u\, x^3 \\ x'^1 = x^1 \\ x'^2 = x^2 \\ x'^3 = \mathrm{sh}\, u\, x^0 + \mathrm{ch}\, u\, x^3 \end{cases} \tag{A.9a}$$

$$\exp(-\mathrm{i}\theta J_3)\begin{cases} x'^0 = x^0 \\ x'^1 = \cos\theta x^1 - \sin\theta x^2 \\ x'^2 = \sin\theta x^1 + \cos\theta x^2 \\ x'^3 = x^3 \end{cases} \tag{A.9b}$$

$$\exp(-\mathrm{i}uI_3)\begin{cases} x'^0 = x^0 \\ x'^1 = \mathrm{ch}\, u\, x^1 - \mathrm{i}\,\mathrm{sh}\, u\, x^2 \\ x'^2 = \mathrm{i}\,\mathrm{sh}\, u\, x^1 + \mathrm{ch}\, u\, x^2 \\ x'^3 = x^3 \end{cases} \tag{A.9c}$$

$$\exp(-\mathrm{i}\theta H_3)\begin{cases} x'^0 = \cos\theta x^0 - \mathrm{i}\sin\theta x^3 \\ x'^1 = x^1 \\ x'^2 = x^2 \\ x'^3 = -\mathrm{i}\sin\theta x^0 + \cos\theta x^3. \end{cases} \tag{A.9d}$$

Appendix 2. Helicity spinors

Massive helicity spinors are obtained from rest-frame solutions of the Dirac equation

$$\omega^1(0) = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad \omega^2(0) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} \qquad \omega^3(0) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} \qquad \omega^4(0) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

by application of a Lorentz transformation S composed of a boost along the three-direction followed by a rotation around an axis perpendicular to the plane defined by the three-axis and the final momentum $\boldsymbol{p}$:

$$S = \frac{1}{[2|p|(p^3+|p|)]^{1/2}}[p^3+|p| - \mathrm{i}(-p^2\sigma^{23} + p^1\sigma^{31})]\left(\frac{p^0+m}{2m}\right)^{1/2}\left(1 - \mathrm{i}\frac{|p|}{p^0+m}\sigma^{03}\right)$$

$$= \frac{1}{2}\left(\frac{p^0+m}{m|p|(p^3+|p|)}\right)^{1/2} \left[\begin{array}{cccc} p^3+|p| & -p_- & \dfrac{(p^3+|p|)|p|}{p^0+m} & \dfrac{p_-|p|}{p^0+m} \\ p_+ & p^3+|p| & \dfrac{p_+|p|}{p^0+m} & -\dfrac{|p|(p^3+|p|)}{p^0+m} \\ \dfrac{|p|(p^3+|p|)}{p^0+m} & \dfrac{p_-|p|}{p^0+m} & p^3+|p| & -p_- \\ \dfrac{p_+|p|}{p^0+m} & -\dfrac{|p|(p^3+|p|)}{p^0+m} & p_+ & p^3+|p| \\ u(p+) & u(p-) & v(p-) & v(p+) \end{array}\right]$$

where $p_\pm = p^1 \pm \mathrm{i}p^2$. The helicity spinors $u(p\mu)$ and $v(p\mu)$ are, as indicated, the columns of this matrix. They satisfy the Dirac equations

$$(\not{p} - m)u(p\mu) = 0$$

$$(\not{p} + m)v(p\mu) = 0$$

and are eigenvectors of the helicity operator $\mathbf{h}$:

$$\mathbf{h} = \tfrac{1}{2}\boldsymbol{p}\,.\,\boldsymbol{\Sigma}/|p| \qquad \sigma^{ij} \equiv \Sigma^k = \begin{bmatrix} \sigma^k & \\ & \sigma^k \end{bmatrix}$$

$$\mathbf{h}u(p+) = \tfrac{1}{2}u(p+) \qquad \mathbf{h}v(p-) = \tfrac{1}{2}v(p-)$$

$$\mathbf{h}u(p-) = -\tfrac{1}{2}u(p-) \qquad \mathbf{h}v(p+) = -\tfrac{1}{2}v(p+).$$

Their normalisation and orthogonality properties are

$$\bar{u}(p\mu)u(p\mu') = \delta_{\mu\mu'} = -\bar{v}(p\mu)v(p\mu')$$

$$u^+(p\mu)u(p\mu') = \frac{p^0}{m}\delta_{\mu\mu'} = v^+(p\mu)v(p\mu').$$

Other useful properties used in the text:

$$\gamma^5 u(p\mu) = v(p-\mu) \qquad \gamma^5 v(p\mu) = u(p-\mu)$$

$$\begin{cases} \gamma^0 u(p\mu) = u(-\boldsymbol{p}-\mu)\exp(-\mathrm{i}\theta(p\mu)) & \theta(p+) = -\arg p_+ \\ \gamma^0 v(p\mu) = -v(-\boldsymbol{p}-\mu)\exp(-\mathrm{i}\theta(p-\mu)) & \theta(p_-) = -\arg(-p_-) \end{cases}$$

$$\begin{cases} \mathrm{i}\gamma^2 u(p\mu) = v^*(p\mu)(-1)^{\epsilon(\mu)} & \epsilon(+) = 0 \\ \mathrm{i}\gamma^2 v(p\mu) = u^*(p\mu)(-1)^{\epsilon(\mu)} & \epsilon(-) = 1 \end{cases}$$

$$\begin{cases} \mathrm{i}\gamma^1\gamma^3 u^*(p\mu) = \mathrm{i}u(-\boldsymbol{p}\mu)(-1)^{\epsilon(-\mu)}\exp(-\mathrm{i}\theta(p-\mu)) \\ \mathrm{i}\gamma^1\gamma^3 v^*(p\mu) = -\mathrm{i}v(-\boldsymbol{p}\mu)(-1)^{\epsilon(-\mu)}\exp(-\mathrm{i}\theta(p\mu)). \end{cases}$$

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